CS548 Advanced Information Security

An Improved Algorithm for Computing Logarithms over GF(p) and Its Cryptographic Significance Function

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What's the problem ?

Pair of Inverse Functions

$$y \equiv \alpha^x \quad (\mathrm{mod}\ p)$$

$$x \equiv \log_\alpha y \quad \text{over} \quad GF(p)$$

easy

difficult

where p is prime, α is a fixed primitive element of $GF(p)$

① $y \equiv \alpha^x\ (\mathrm{mod}\ p)$: $O(\log_2 p)$ time complexity (ex. $\alpha^{18} = (((\alpha^2)^2)^2)^2\alpha^2$)

② $x \equiv \log_\alpha y$ over $GF(p)$: Previously , $O(\sqrt{p})$ time & space complexity

One-way Function : Original problem – easy
Inverse problem – difficult

Really ?

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p-1 Must Have Large Prime Factor !

$$x \equiv log_{\alpha} y \quad over \quad GF(p)$$

- ✓ Can we solve this problem faster than $O(\sqrt{p})$ time ??

- ✓ Over GF(p), when **<u>p-1 has only small prime factors</u>**,

  the logarithm problem can be solved $O(log^2 p)$

- ✓ To make one-way function,
  <u>p-1 must have at least one large prime factor</u>

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Use in Cryptography

- For plain-text M, key K, cipher-text C with the restrictions

  $1 \le M, C \le p-1, \quad 1 \le K \le p-2, \quad GCD(K, p-1) = 1,$

  $$\boldsymbol{C \equiv M^K \ (mod\ \ p)}$$

- For deciphering operation,

  $$\boldsymbol{M \equiv C^D \ (mod\ \ p)}, \quad \text{where} \quad D \equiv K^{-1} \ (\bmod p-1)$$

  (*D* is uniquely determined because *GCD(K, p-1) = 1*)

- Finding the key K is equivalent to computing

  $$\boldsymbol{K \equiv log_M C} \quad \text{over } \boldsymbol{GF(p)}$$

Background – Finite Field (Algebra)

✓ ***GF(p)*** : Galois Field (a.k.a. Finite Field)

➔ A field that contains only finitely many elements

✓ Computations over *GF(p)*

ex. When *p=5* (i.e. *GF(5)*),

$$3 + 4 \equiv 2 \pmod{5}, \quad 3 - 4 \equiv 4 \pmod{5}$$
$$3^2 = 9 \equiv 4 \pmod{5}, \quad \log_3 4 \equiv 2 \pmod{5}$$

✓ **Primitive Element** : A generator of the multiplicative group of the field

ex. $3^1 \equiv 3, \quad 3^2 \equiv 4, \quad 3^3 \equiv 2, \quad 3^4 \equiv 1 \pmod{5}$

So, 3 is a primitive element of *GF(5)*

Background - Number Theory (1)

- **Euler's φ - function** (a.k.a. Euler's totient function)

$$\varphi(p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}) = (p_1 - 1) p_1^{e_1} (p_2 - 1) p_2^{e_2} \cdots (p_k - 1) p_k^{e_k} \;, \quad \text{where } p_i\text{'s are prime}$$

$$= p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} \prod_{p_i} \left(1 - \frac{1}{p_i}\right)$$

- **The fraction ρ**

$$\rho = \frac{\varphi(p-1)}{p-1} = \prod_{p_i | (p-1)} \left(1 - \frac{1}{p_i}\right) \qquad \left(\forall p < 1.6 \times 10^{103} \Rightarrow p > 0.1\right)$$

- For primes of the form $p = 2p' + 1$, with p' prime, $\rho = \frac{1}{2}\left(1 - \frac{1}{p'}\right) \approx \frac{1}{2}$

Background - Number Theory (2)

- **Fermat's Little Theorem**

  $$z^{p-1} \equiv 1 \pmod{p}\ ,\quad 1 \le z \le p-1$$

- From the theorem

  $$z^{x} \equiv z^{x \,(\mathrm{mod}\, p-1)} \pmod{p}$$

- **Chinese Remainder Theorem**

  Suppose that $n_1, n_2, \ldots, n_k$ are positive integers which are pairwise coprime.

  For any integers $a_1, a_2, \ldots, a_k$, there exist an integer x (mod $n_1 n_2 \ldots n_k$) satisfying

  $$x \equiv a_1 \pmod{n_1}\ ,\quad x \equiv a_2 \pmod{n_2}\ , \cdots,\ x \equiv a_k \pmod{n_k}$$

- c.f. **Euler's Theorem**

  For any positive integer n, z ($GCD(n,z)=1$)

  $$z^{\varphi(n)} \equiv 1 \pmod{n}$$

An Algorithm for p = 2^n+1 (1)

- Given $\alpha,\ p,\ y$, (α is a primitive element of *GF(p)*)

  Must find *x* such that $y \equiv \alpha^x \pmod{p}$

- Let $x = \sum_{i=0}^{n-1} b_i 2^i$,

- Then, b_0 is determined by

$$y^{(p-1)/2} \pmod{p} = \begin{cases} +1\,, & \text{if } b_0 = 0 \\ -1\,, & \text{if } b_0 = 1 \end{cases}$$

($\because$) Since α is primitive, $\alpha^{(p-1)/2} \equiv -1 \pmod{p}$

Therefore, $y^{(p-1)/2} = (\alpha^x)^{(p-1)/2} \equiv (\alpha^{(p-1)/2})^x \pmod{p}$

An Algorithm for p = 2^n+1 (2)

- Now, b_1 is determined by letting

  $$z \equiv y\alpha^{-b_0} \equiv \alpha^{x_1} \pmod{p}, \quad \text{where } x_1 = \sum_{i=1}^{n-1} b_i 2^i$$

- Then,

  $$z^{(p-1)/4} \pmod{p} \equiv (\alpha^{x_1})^{(p-1)/4} \equiv (\alpha^{(p-1)/2})^{x_1/2} \equiv (-1)^{x_1/2}$$

  $$\equiv \begin{cases} +1, & b_1 = 0 \\ -1, & b_1 = 1 \end{cases}$$

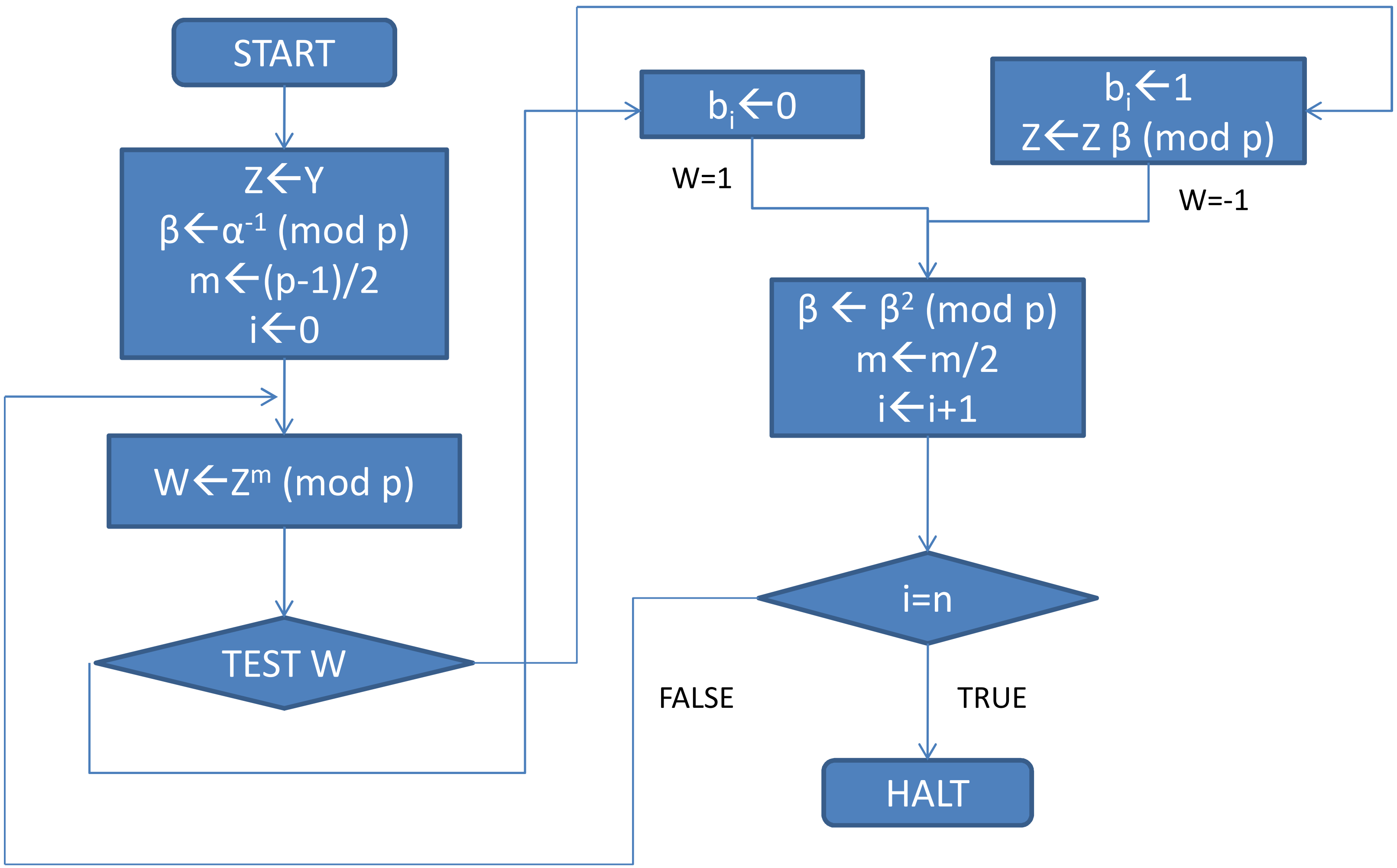
- Remaining bit of *x*

  $$m \equiv {p-1}\Big/{2^{i+1}}$$

  $$z \equiv \alpha^{x_i} \pmod{p}, \quad \text{where } x_i = \sum_{j=i}^{n-1} b_j 2^j$$

  $$z^m \pmod{p} \equiv \begin{cases} +1, & b_i = 0 \\ -1, & b_i = 1 \end{cases}$$

Flowchart for p = 2^n+1



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An Algorithm for Arbitrary Primes (1)

- Generalize the algorithm to arbitrary primes *p*
  - $2^{16}+1$ is the largest known prime of the form 2^n+1

- Let $p-1 = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, $p_i < p_{i+1}$

  where the p_i are distinct primes and the n_i are positive integers

- By Chinese Remainder Theorem,

  if the value of $x(\bmod p_i^{n_i})$ is determined for all *i*, then

  $x \ (\bmod \prod_{i=1}^{k} p_i^{n_i}) = x(\bmod p-1) = x$

An Algorithm for Arbitrary Primes (2)

- Consider the following expansion of $x \pmod{p_i^{n_i}}$

$$x \pmod{p_i^{n_i}} = \sum_{j=0}^{n_i - 1} b_j p_i^{\,j} \;, \quad \text{where } 0 \le b_j \le p_i - 1$$

- Then,

$$y^{(p-1)/p_i} \equiv \alpha^{(p-1)x/p_i} \equiv \gamma_i^{\,x} \equiv \gamma_i^{\,b_0} \pmod{p}$$

$$\text{where } \gamma_i = \alpha^{(p-1)/p_i}$$

→ The resultant value uniquely determines b_0

An Algorithm for Arbitrary Primes (3)

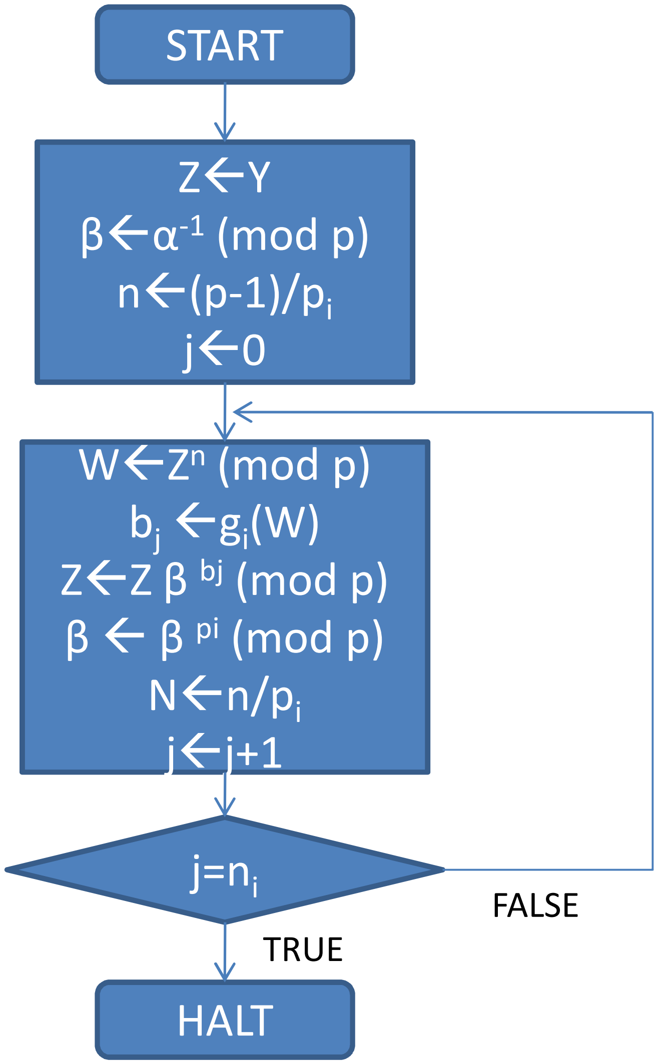
- ✓ The function $g_i(w)$ is defined by

$$\gamma_i^{\,g_i(w)} \equiv w \pmod{p}\,, \quad 0 \le g_i(w) \le p_i - 1$$

- ✓ The resultant value, $y^{(p-1)/p_i} \equiv \gamma_i^{\,b_0} \pmod{p}$ determines b_i by $g_i(w)$

- ✓ So, dominant computational requirement : **computing** $\boldsymbol{g_i(w)}$

Flowchart for arbitrary primes



Time & Space Complexity

Theorem

Let

$$p-1 = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}, \qquad p_i < p_{i+1}$$

be the prime factorization of $p-1$, where p is prime, the p_i are distinct primes, and the n_i are positive integer.

Then, for any $\{r_i\}_{i=1}^{k}$ with all $0 \le r_i \le 1$, logarithms over *GF(p)* can be computed in $O\left(\sum_{i=1}^{k} n_i (\log_2 p + p_i^{1-r_i}(1+\log_2 p_i^{r_i}))\right)$ operations with $O\left(\log_2 p \cdot \sum_{i=1}^{k}(1+p_i^{r_i})\right)$ bits of memory.

Proof. [1]

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Discussion

- ✓ $p = 2^{448} \cdot 5^2 + 1$ is a prime that *p-1* has only small prime factors (i.e., 2, 5)
  - → 2+448 = 450 iterations of the loop in the flowchart
  - **Not one-way function**
  - → Dominant computational requirement: <u>450 exponentiations *mod p*</u>

- ✓ When $p = 2p' + 1$, (*p'* is also prime)

  (ex. $p' = 2^{121} \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 + 1$)
  - → Dominant computational requirement: $g(w)$
  - **One-way function**
  - → computing *g(w)*: <u>more than 10^{30} operations & 10^{30} bits of memory</u> (*r=1/2*)

References

[1] S. C. Pohlig and M. E. Hellman, An Improved Algorithm for Computing Logarithms over GF(p) and Its Cryptographic Significance, IEEE Transactions on Information Theory, 1978